

Solution Set 4 (compiled by Daniel Larson)

1. **Griffiths 3.33** To get a general formula for the electric field from an electric dipole, let's start with the general formula for the potential, (3.99) in the text.

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = -\nabla V_{\text{dip}}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \nabla \left(\frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} \right) = -\frac{1}{4\pi\epsilon_0} \nabla \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right) = -\frac{1}{4\pi\epsilon_0} \left[\frac{1}{r^3} \nabla(\mathbf{p} \cdot \mathbf{r}) + (\mathbf{p} \cdot \mathbf{r}) \nabla \left(\frac{1}{r^3} \right) \right]$$

To evaluate the first term we use some vector identities.

$$\nabla(\mathbf{p} \cdot \mathbf{r}) = \mathbf{p} \times (\nabla \times \mathbf{r}) + \mathbf{r} \times (\nabla \times \mathbf{p}) + (\mathbf{p} \cdot \nabla) \mathbf{r} + (\mathbf{r} \cdot \nabla) \mathbf{p} = (\mathbf{p} \cdot \nabla) \mathbf{r},$$

because $\nabla \times \mathbf{r} = 0$ and \mathbf{p} is a constant vector, so any derivatives of it vanish. To evaluate the one remaining term we can temporarily choose cartesian coordinates:

$$(\mathbf{p} \cdot \nabla) \mathbf{r} = \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z} \right) (x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}) = p_x \hat{\mathbf{x}} + p_y \hat{\mathbf{y}} + p_z \hat{\mathbf{z}} = \mathbf{p}.$$

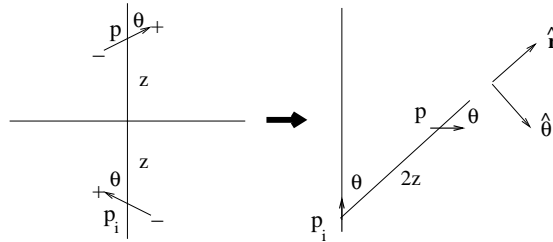
For the second term we need $\nabla r^n = n r^{n-1} \hat{\mathbf{r}}$. Putting the results together,

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \left[\frac{1}{r^3} \mathbf{p} + (\mathbf{p} \cdot \hat{\mathbf{r}}) r \left(\frac{-3}{r^4} \right) \hat{\mathbf{r}} \right] = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}]$$

2. **Griffiths 4.4** The point charge produces an electric field with magnitude $E_1 = q/4\pi\epsilon_0 r^2$ at the location of the neutral atom. That electric field polarizes the atom, giving it a dipole moment $\mathbf{p} = \alpha \mathbf{E}_1 = -\alpha q/4\pi\epsilon_0 r^2 \hat{\mathbf{r}}$ where \mathbf{r} is the vector pointing from the atom towards the point charge. But the polarized atom produces its own field due to its dipole moment \mathbf{p} . At the location of the point charge, the electric field is $\mathbf{E}_2 = 2\mathbf{p}/4\pi\epsilon_0 r^3$ where I've used the result of the previous problem. Finally, the force felt by the point charge is attractive:

$$\mathbf{F} = q\mathbf{E}_2 = -\frac{\alpha q^2}{8\pi^2 \epsilon_0^2 r^5} \hat{\mathbf{r}} = -2\alpha \left(\frac{q}{4\pi\epsilon_0} \right)^2 \frac{1}{r^5} \hat{\mathbf{r}}.$$

3. **Griffiths 4.6** To determine the effect of the conducting plane, we can use an image dipole situated below the plane. To figure out how it should be pointing, we can think of the perfect dipole as two charges separated by a small distance, figure out where the image charges should be, and then let the distance between the charges in each dipole go to zero. This is shown in the figure.



The image dipole, \mathbf{p}_i , creates an electric field \mathbf{E}_i at the position of the real dipole, which causes a torque on the real dipole, $\mathbf{N} = \mathbf{p} \times \mathbf{E}_i$. If we choose a coordinate system centered on the image dipole with \mathbf{p}_i pointing in the z -direction, then the real dipole can be taken to be in the xz -plane with coordinates $(r, \theta, \phi) = (2z, \theta, 0)$. Using equation (3.103) in the text, the electric field there is $\mathbf{E}_1 = \frac{p_i}{4\pi\epsilon_0 (2z)^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$. Now, in order to take the cross product, we need to resolve \mathbf{p} in the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ directions. From the figure we see that \mathbf{p} makes an angle θ with the $\hat{\mathbf{r}}$ direction. Thus $\mathbf{p} = p \cos \theta \hat{\mathbf{r}} + p \sin \theta \hat{\boldsymbol{\theta}}$.

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}_i = (p \cos \theta \hat{\mathbf{r}} + p \sin \theta \hat{\boldsymbol{\theta}}) \times \frac{p_i}{4\pi\epsilon_0 (2z)^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}) = \frac{-p^2 \cos \theta \sin \theta}{4\pi\epsilon_0 (2z)^3} \hat{\boldsymbol{\phi}} = -\frac{1}{4\pi\epsilon_0} \frac{p^2 \sin(2\theta)}{16z^3} \hat{\boldsymbol{\phi}}$$

where we have used $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ and $p_i = p$. Note that $-\hat{\boldsymbol{\phi}}$ is the direction out of the page. The torque vanishes for $\theta = 0, \pi/2$, and π . However, since the dipole wants to rotate one way for $0 < \theta < \pi/2$ and the other way for $\pi/2 < \theta < \pi$, at $\pi/2$ the torque is changing sign and so the dipole is not stable at that angle. The stable orientations are for $\theta = 0$ or π where the dipole is perpendicular to the conducting plane, pointing either toward or away from it.

4. Griffiths 4.10

- (a) $\sigma_b = \mathbf{P}(R) \cdot \hat{\mathbf{n}} = kR \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = kR$. $\rho_b = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 k r) = -\frac{1}{r^2} 3kr^2 = -3k$.
- (b) There is no free charge specified, and the sphere is not connected to any wires or batteries, so the free charge is zero everywhere. Thus the only charges contributing to the electric field are the bound charges. Because of the symmetry we know that the electric field can only be pointing radially, so we can use Gauss's law to find the field. Inside the sphere we make a gaussian sphere of radius r : $E(r)4\pi r^2 = \frac{4}{3}\pi r^3 \rho_b / \epsilon_0 \Rightarrow \mathbf{E}_{\text{in}}(r) = \rho_b r / 3\epsilon_0 \hat{\mathbf{r}} = -k/\epsilon_0 \mathbf{r}$. Outside the sphere the total volume charge is $-3k \frac{4}{3}\pi R^3 = -4\pi R^3 k$ while the total surface charge is $kR \times 4\pi R^2 = 4\pi R^3 k$. Thus the net charge inside a Gaussian surface with radius $r > R$ is zero, which means $\mathbf{E}_{\text{out}} = \mathbf{0}$.

5. **Griffiths 4.13** We want to tackle this problem in exactly the same way we did the sphere with uniform polarization in class, or in example 4.3 in the text. We can think of the uniformly polarized cylinder as two cylinders with opposite uniform charge density $\pm\rho$ separated from each other by a small distance d . Start by considering a single, uniformly charged cylinder. Using Gauss's law we can find the electric field both outside and inside the cylinder. Using Griffiths's notation with \mathbf{s} as the radial coordinate, inside the cylinder we find: $E2\pi s\ell = \frac{1}{\epsilon_0}\rho\pi s^2\ell \Rightarrow \mathbf{E} = (\rho/2\epsilon_0)\mathbf{s}$. In the region of overlap between the two cylinders we have contributions to \mathbf{E} from both cylinders which add just like in Problem 2.18 (see solution set 2 for a figure). *However*, in this case we'll define \mathbf{d} to be the vector pointing from the center of the negative cylinder to the center of the positive cylinder. Thus the total electric field in the region of overlap is $\mathbf{E} = -(\rho/2\epsilon_0)\mathbf{d}$. We can think of the two uniformly charged cylinders as being line charges with charge per length $\pm\lambda = \pm\pi a^2\rho$, which is like a bunch of dipoles $\lambda d\ell\mathbf{d}$ all in a row. Now thinking back to the single, polarized cylinder, the total dipole moment in a piece of length ℓ is $\mathbf{P}(\pi a^2\ell) = \lambda\ell\mathbf{d} = \pi a^2\rho\ell\mathbf{d}$, so $\mathbf{P} = \rho\mathbf{d}$. Plugging this into our expression for \mathbf{E} we find $\mathbf{E}_{\text{in}} = -\frac{1}{2\epsilon_0}\mathbf{P}$.

Now we need the electric field outside the cylinders. This time, for a single uniformly charged cylinder and $s > a$ Gauss's law gives: $E2\pi s\ell = \frac{1}{\epsilon_0}\pi a^2\ell\rho \Rightarrow \mathbf{E} = (\rho a^2/2\epsilon_0 s)\hat{\mathbf{s}}$. At some point outside the cylinders, let \mathbf{s}_+ and \mathbf{s}_- be the radial vectors from the centers of the two charged cylinders to the point in question. The total electric field at that point gets contributions from both cylinders, so

$$\mathbf{E}_{\text{out}} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho a^2}{2\epsilon_0} \left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-} \right) = \frac{\rho a^2}{2\epsilon_0} \left(\frac{\mathbf{s}_+}{s_+^2} - \frac{\mathbf{s}_-}{s_-^2} \right)$$

We want to simplify this expression, using the fact that $\mathbf{s}_+ - \mathbf{s}_- = -\mathbf{d}$ and $d \ll s_+, s_-$. Let \mathbf{s} be the radial vector from the midpoint between the two charged cylinders; this is the true center of the uniformly polarized cylinder. Then $\mathbf{s}_{\pm} = \mathbf{s} \mp \frac{\mathbf{d}}{2}$.

$$\begin{aligned} \frac{\mathbf{s}_{\pm}}{s_{\pm}^2} &= \left(\mathbf{s} \mp \frac{\mathbf{d}}{2} \right) \left(s^2 \mp \mathbf{s} \cdot \mathbf{d} + \frac{d^2}{4} \right)^{-1} = \left(\mathbf{s} \mp \frac{\mathbf{d}}{2} \right) \frac{1}{s^2} \left(1 \mp \frac{\mathbf{s} \cdot \mathbf{d}}{s^2} + \frac{d^2}{4s^2} \right)^{-1} \\ &\approx \frac{1}{s^2} \left(\mathbf{s} \mp \frac{\mathbf{d}}{2} \right) \left(1 \pm \frac{\mathbf{s} \cdot \mathbf{d}}{s^2} \right) = \frac{1}{s^2} \left(\mathbf{s} \pm \mathbf{s} \frac{\mathbf{s} \cdot \mathbf{d}}{s^2} \mp \frac{\mathbf{d}}{2} \right) \end{aligned}$$

where we have kept only the terms linear in the small quantity d/s . Using this result in the expression for the electric field, and the result that $\mathbf{P} = \rho\mathbf{d}$, we find

$$\mathbf{E}_{\text{out}} = \frac{\rho a^2}{2\epsilon_0} \frac{1}{s^2} \left[\left(\mathbf{s} + \mathbf{s} \frac{\mathbf{s} \cdot \mathbf{d}}{s^2} - \frac{\mathbf{d}}{2} \right) - \left(\mathbf{s} - \mathbf{s} \frac{\mathbf{s} \cdot \mathbf{d}}{s^2} + \frac{\mathbf{d}}{2} \right) \right] = \frac{\rho a^2}{2\epsilon_0} \frac{1}{s^2} \left(\frac{2\mathbf{s}(\mathbf{s} \cdot \mathbf{d})}{s^2} - \mathbf{d} \right) = \frac{a^2}{2\epsilon_0 s^2} [2(\mathbf{P} \cdot \hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}]$$

6. **Griffiths 4.15**

(a)

$$\rho_b = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{k}{r} \right) = -\frac{k}{r^2}; \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \mathbf{P} \cdot \hat{\mathbf{r}} = k/b & (\text{at } r = b), \\ \mathbf{P} \cdot (-\hat{\mathbf{r}}) = -k/a & (\text{at } r = a). \end{cases}$$

The spherical symmetry again tells us that \mathbf{E} is radial. Thus $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q_{\text{enc}}}{r^2} \hat{\mathbf{r}}$ in all three regions. For a gaussian surface with $r < a$ there is no charge enclosed, so $\mathbf{E}(r < a) = 0$. For $a < r < b$ the charge enclosed is $(\frac{k}{a})4\pi a^2 + \int_a^r (\frac{-k}{r'^2})4\pi r'^2 dr' = -4\pi k a - 4\pi k(r - a) = -4\pi k r$, so $\mathbf{E}(a < r < b) = -(k/\epsilon_0 r) \hat{\mathbf{r}}$. Finally, for $r > b$ the charge inclosed is the same as in the previous calculation (with $r = b$) plus the surface charge at $r = b$. So $Q_{\text{enc}} = -4\pi k b + 4\pi b^2(k/b) = 0$, thus $\mathbf{E}(r > b) = 0$.

- (b) The spherical symmetry tells us \mathbf{D} must be radial, so $\int \mathbf{D} \cdot d\mathbf{a} = 4\pi r^2 D(r)$ at some radius r . But since there is no free charge anywhere, we must have $\text{div } \mathbf{D} = 0$ everywhere. Since $\epsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P} = -\mathbf{P}$, since $\mathbf{P} = 0$ both inside and outside the shell, $\mathbf{E} = 0$ both outside and inside the shell. Within the shell, $\mathbf{E} = -P/\epsilon_0 = -(k/\epsilon_0 r) \hat{\mathbf{r}}$. This agrees with part (a) and was far quicker.

7. **Griffiths 4.16** We want to find \mathbf{D} and \mathbf{E} inside the cavity. This is easiest to do by considering the superposition of the original piece of polarized dielectric without a hole and a piece of dielectric in the shape of the cavity possessing opposite polarization. The the fields at the center of the cavity will be the sum of the fields due to the original dielectric (namely \mathbf{E}_0 and \mathbf{D}_0) with the fields at the center of uniformly polarized objects in the shape of the cavity, which we can call \mathbf{E}' and \mathbf{D}' . It is the latter fields that we must determine.

- (a) The fields at the center of a uniformly polarized sphere were found in example 4.3. If \mathbf{P} is the polarization of the original dielectric with a cavity, then $-\mathbf{P}$ is the polarization of the cavity-shaped piece we are superimposing. So $\mathbf{E}' = -\frac{1}{3\epsilon_0}(-\mathbf{P})$. Thus the polarization in the cavity is $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}' = \mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{P}$. Also, since the polarization in the cavity is zero, we have $\mathbf{D} = \epsilon_0 \mathbf{E} = \epsilon_0 \mathbf{E}_0 + \frac{1}{3}\mathbf{P} = \mathbf{D}_0 - \mathbf{P} + \frac{1}{3}\mathbf{P} = \mathbf{D}_0 - \frac{2}{3}\mathbf{P}$.
- (b) A long thin needle with polarization $-\mathbf{P}$ looks like a bunch of dipoles sitting end to end in a long line, like Figure 4.11 in the text. Thus the net charge that contributes to the electric field at the center of the needle are positive and negative charges on the ends of the needle. But for a very long and very thin needle these will be small charges and far away, so will have negligible contribution. Thus $\mathbf{E}' = 0$, so $\mathbf{E} = \mathbf{E}_0$. Again, in the cavity there is no polarization so $\mathbf{D} = \epsilon_0 \mathbf{E} = \epsilon_0 \mathbf{E}_0 = \mathbf{D}_0 - \mathbf{P}$.
- (c) The thin wafer shape has the field of a parallel plate capacitor with charge $\sigma_b = \mathbf{P}' \cdot \hat{\mathbf{n}} = -P$ on the upper plate and the opposite charge on the bottom plate. The electric field between the plates is then pointing up, in the same direction as \mathbf{P} , and has magnitude P/ϵ_0 . Thus $\mathbf{E}' = \frac{1}{\epsilon_0}\mathbf{P}$, so $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}' = \mathbf{E}_0 + \frac{1}{\epsilon_0}\mathbf{P}$. Finally, $\mathbf{D} = \epsilon_0 \mathbf{E} = \epsilon_0 \mathbf{E}_0 + \mathbf{P} = \mathbf{D}_0$.

8. **Griffiths 4.18** Choose coordinates so that the capacitor is in the xy -plane and $\hat{\mathbf{z}}$ points “up” from the negatively charged plate towards the positively charged plate. Let’s start this problem by thinking physically about what will happen. There is free charge placed on the top and bottom plates, which will produce some electric field pointing down. (We will assume the capacitor is big enough in the xy -directions so that the electric field will be only in the z -direction.) But that electric field will polarize the two dielectrics, producing \mathbf{P} pointing in the same direction as \mathbf{E} , which in turn induces positive bound surface charge on the bottoms of each dielectric surface and negative bound charge on the top of each dielectric surface. These collections of bound charge will also produce their own electric field, which we also need to take into account. Now let’s work through the details.

- (a) The \mathbf{D} field depends only on the free charge, so with $+\sigma$ on the top plate and $-\sigma$ on the bottom plate, the \mathbf{D} field in between the plates will be $\mathbf{D} = \sigma(-\hat{\mathbf{z}})$, which is the \mathbf{D} field between two infinite planes with free surface charges $\pm\sigma$. It has the same value in each of the slabs.

- (b) Since we're dealing with linear dielectrics, $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$. However, ϵ_r is different in the two slabs. Thus in slab 1, $\mathbf{E}_1 = \mathbf{D}/\epsilon_0 \epsilon_r^{(1)} = -\sigma/2\epsilon_0 \hat{\mathbf{z}}$, and in slab 2, $\mathbf{E}_2 = \mathbf{D}/\epsilon_0 \epsilon_r^{(2)} = -\sigma/1.5\epsilon_0 \hat{\mathbf{z}} = -2\sigma/3\epsilon_0 \hat{\mathbf{z}}$.
- (c) In a linear dielectric, $\mathbf{P} = \epsilon_0(\epsilon_r - 1)\mathbf{E}$, so in slab 1 we have $\mathbf{P}_1 = \epsilon_0(2 - 1)\mathbf{E}_1 = -\sigma/2 \hat{\mathbf{z}}$ and in slab 2 we have $\mathbf{P}_2 = \epsilon_0(1.5 - 1)\mathbf{E}_2 = -\sigma/3 \hat{\mathbf{z}}$.
- (d) We find the potential difference by integrating $\mathbf{E} \cdot d\ell$ between the two plates. Since \mathbf{E} is uniform in each of the slabs, and points straight down, we get $V = E_1 a + E_2 a = 7a\sigma/6\epsilon_0$.
- (e) $\rho_b = -\nabla \cdot \mathbf{P} = 0$ in both slabs. $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$, so remembering that $\hat{\mathbf{n}}$ always points out of each slab, $\sigma_b^{(1)} = +\sigma/2$ at the bottom of slab 1 and minus that at the top of slab 1. $\sigma_b^{(2)} = +\sigma/3$ at the bottom of slab 2 and minus that at the top. See the figure below.
- (f) Using all of the charges, we want to recalculate the electric field in each slab. All of the charges are surface charges distributed on (approximately) infinite planes of charge, so we use the result that the electric field due to a single plane of surface charge doesn't depend on the distance from the plane. Inside slab 1 it is as if there was a single plane on top with net surface charge equal to the free charge on the top capacitor plate plus the bound charge on the top of slab 1, namely $\sigma - \sigma/2 = \sigma/2$; and also a single plane below with net surface charge due to the bound charge on the bottom of slab 1, the top of slab 2, and the bottom of slab 2, and the free charge on the bottom capacitor plate, namely $\sigma/2 - \sigma/3 + \sigma/3 - \sigma = -\sigma/2$. So inside slab 1 the electric field is the same as between two infinite plates with surface charge $\pm\sigma/2$, so $E_1 = \sigma/2\epsilon_0$ (pointing down). Similarly, inside slab 2 there is net surface charge $\sigma - \sigma/2 + \sigma/2 - \sigma/3 = 2\sigma/3$ above and $\sigma/3 - \sigma = -2\sigma/3$ below. So the electric field in slab 2 is $E_2 = 2\sigma/3\epsilon_0$, again pointing down.



Problem 8. Griffiths 3.18